Lossless Data Compression on Unbounded Integers

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Objectives

- Formulate compression algorithms that can be applied to the set of unbounded integers using symbol compression techniques
- Prove the universality and asymptotic optimality of these algorithms
- Demonstrate the practical advantages of using these algorithms on sets of integers

Definitions

Definition

A *lossless data compression algorithm* is one that encodes input into codewords that requires less storage and loses no information.

Definition

Entropy is a theoretical lower bound on the performance of a lossless compression algorithm. It is given by a function of the PMF (probability mass function) as shown:

$$H(P) = -\sum_i P_i \log P_i$$

Definitions

Definition

A *fixed-length code* (FLC) is one that uses a fixed number of bits to represent a symbol or integer - even if some bits are not necessary, they will be padded.

Definition

A *variable-length code* (VLC) is one that uses a variable number of bits to represent a symbol or integer, following the heuristic that higher-probability symbols have shorter codes and vice versa.

Definitions

Definition

A code is *universal* if the average code length is bounded by a multiple of the entropy, so

$$\frac{E_P(L_\rho)}{\max\{1, H(P)\}} \leq K_\rho.$$

Definition

A code is *asymptotically optimal* if the ratio between the average code length and the entropy is bounded by a function of entropy that converges to 1, so

$$\frac{E_P(L_\rho)}{\max\{1, H(P)\}} \leq R_\rho(H(P)) \leq K_\rho$$

with

$$\lim_{H \to \infty} R_\rho(H) = 1.$$

Definitions

Definition

In *Elias*-γ coding for integers, if the length of the binary representation of an integer x is N bits, then we prepend $N-1$ zeroes with the binary representation of that integer.

$$17 = \underbrace{0\ 0\ 0\ 0}_{\text{N-1 zeroes}}\ \overbrace{1\ 0\ 0\ 0\ 1}^{\text{x in binary}}$$

Definitions

Definition

In *Elias-δ* coding for integers, if the length of the binary representation of an integer x is N bits, then we prepend the Elias-γ encoding of N to the last $N-1$ bits of the binary representation of x.

$$17 = \underbrace{0\ 0\ 1\ 0\ 1}_{\gamma(N)} \overbrace{0\ 0\ 0\ 1}^{N-1 \text{ bits}}$$

Definitions

Definition

In *Shannon-Fano-Elias (SFE)* coding for a fixed set of symbols, where a source probability distribution is known, codewords for symbols are generated as the first $\left\lceil \log_2 \frac{1}{p(x)} \right\rceil + 1$ bits to the right of the decimal point in the binary form of

$$P(x) = \sum_{x_0 < x} P(x_0) + \frac{1}{2} P(x)$$

.

- Useful for encryption compared to other compression techniques due to different codewords for different symbol permutations
- The average code length is bounded between $1 + H(P)$ and $2 + H(P)$
- Only usable when source probability distribution is known

Results

Theorem

Given initially empty set AT (already transmitted), a symbol NYT (not yet transmitted) with constant probability $\frac{1}{n+1}$, where n is the number of symbols processed so far, and the FLC $R(x)$ for the symbol, Adaptive-SFE is defined as follows,

1. *Encode the symbol as $E(x)$, where:*

$$E(x) = \begin{cases} SFE(NYT) + R(x) & x \notin AT \\ SFE(x) & x \in AT \end{cases}$$

2. *Update AT and our probability distribution with the symbol in consideration.*

- Does not require a predetermined PMF

Results

Theorem

Canonical δ-SFE is defined as follows:

$$C\delta\text{-}SFE(x) = \begin{cases} SFE(NYT) + \delta(x) & x \notin AT \\ SFE(x) & x \in AT \end{cases}$$

The probability of the NYT element is kept constant at $\frac{1}{n+1}$ where n is the number of symbols processed.

Theorem

Canonical δ-SFE is universal, but not asymptotically optimal.

Results

Theorem

Increment δ-SFE follows the same encoding scheme of canonical δ-SFE, so

$$I\delta\text{-}SFE(x) = \begin{cases} SFE(NYT) + \delta(x) & x \notin AT \\ SFE(x) & x \in AT \end{cases}$$

However, the probability of the NYT element is incremented by a constant k for every occurrence of a new symbol.

Theorem

Increment δ-SFE is both universal and asymptotically optimal.

Results

Theorem

Flagged δ-SFE uses a binary flag to differentiate between occurrences of new or repeated symbols and is defined as follows:

$$F\delta\text{-}SFE(x) = \begin{cases} \delta(x+1) & x \notin AT \\ 1 + SFE(x) & x \in AT \end{cases}$$

Theorem

Flagged δ-SFE is both universal and asymptotically optimal.

γ-RNS is a combination of the Elias-γ encoding scheme and the Residue Number System.

Theorem

Given an integer j, the primes $p_1 = 2, p_2 = 3, ...p_k$ are used such that k is the minimum integer that satisfies $\prod_{n=1}^{k} p_k > j$. The residues $j \mod p_n$ are then calculated and expressed in binary. To ensure uniform lengths, we prepend 0s to the binary representation of each residue r_n until the length of each binary representation reaches $\lceil \log_2 p_n \rceil$. Then, we prepend $k - 1$ 0s and a 1 to the start of the bitstring.

Results

δ-RNS is a combination of the Elias-δ encoding scheme and the Residue Number System.

Theorem

Taking the binary representation of the residues (generated using the same method as for γ-RNS), we instead prepend the binary representation k of the length of the residues + 1. Then, we prepend the unary representation in 0s of the length of k - 1.

Experimental Setup

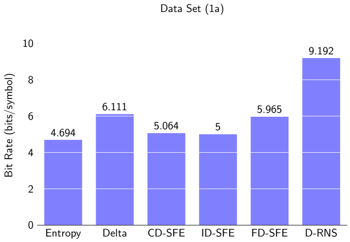
Bit Rate

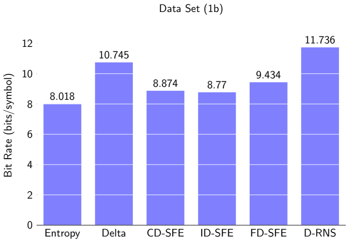
- All data sets consist of 10,000 integers following some distribution
- Data Set 1: geometric PMF distributions with $p = 0.1, 0.01$
- Data Set 2: Poisson distribution with $\lambda = 128$
- Data Set 3: pseudo-randomly generated integers from 1 to 1000
- Compare with entropy and Elias-δ

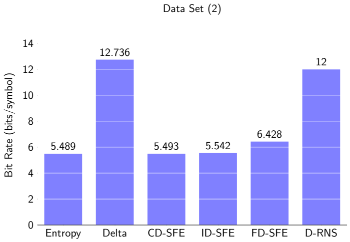
δ-RNS independent testing

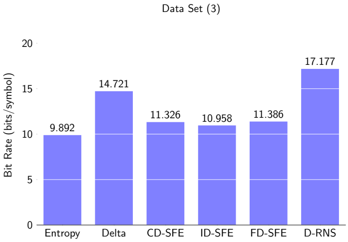
- δ-RNS creates codes of individual symbols
- Depicts performance on integers up to 2^{32}
- Compare with γ-RNS and Elias-δ

Results

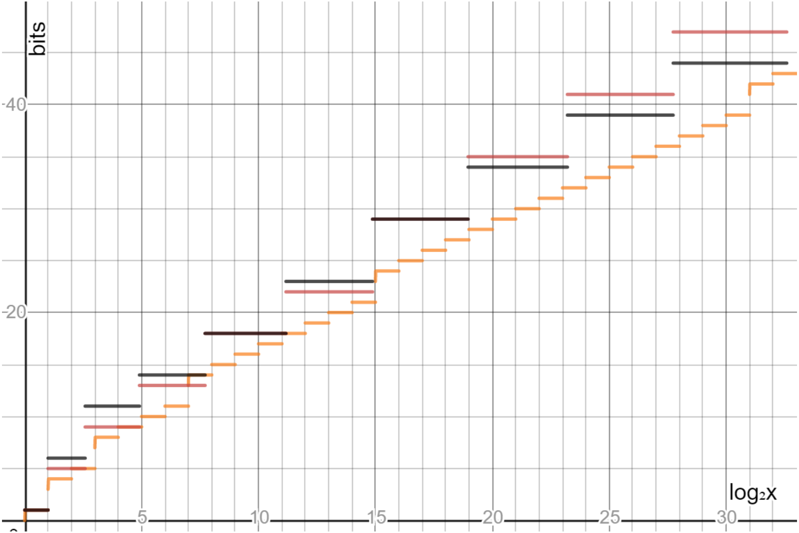

Data Set (1a)
Bit Rate (bits/symbol)
0
2
4
6
8
10
4.694
6.111
5.064
5
5.965
9.192
Entropy
Delta
CD-SFE
ID-SFE
FD-SFE
D-RNS


Data Set (1b)
Bit Rate (bits/symbol)
0
2
4
6
8
10
12
8.018
10.745
8.874
8.77
9.434
11.736
Entropy
Delta
CD-SFE
ID-SFE
FD-SFE
D-RNS


Data Set (2)
Bit Rate (bits/symbol)
0
2
4
6
8
10
12
14
5.489
12.736
5.493
5.542
6.428
12
Entropy
Delta
CD-SFE
ID-SFE
FD-SFE
D-RNS


Data Set (3)
Bit Rate (bits/symbol)
0
5
10
15
20
9.892
14.721
11.326
10.958
11.386
17.177
Entropy
Delta
CD-SFE
ID-SFE
FD-SFE
D-RNS

Results



Conclusion and Future Work

- δ-SFE versions operate at high compression ratios near entropy for unbounded integers
- Applications of prime numbers to compression yield high efficiency gains, especially for large integers
- Investigate similar applications of Elias-δ to other symbol compression techniques such as arithmetic coding
- Applications
  - Compression of inverse indexes used by databases and search engines
  - Extension of super-exponential decryption time of SFE to infinite alphabets
  - Efficient operations on large integers represented in δ-RNS

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